TA NOTES

2.In this problem we show how to generalize Theorem3.3.2(Abel's theorem) to higher order equations. We first outline the procedure for the third order equation

$$y''' + p_1(t)y'' + p_2(t)y' + p_3(t)y = 0$$

Let y_1, y_2, and y_3 be solutions of this equation on an interval I.

(a) If $W = W(y_1, y_2, y_3)$, show that

$$W' = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}.$$

Hint: The derivative o fa 3-by-3 determinant is the sum of three 3-by-3 determinants obtained by differentiating the first, second,and third rows, respectively.

(b) Substitute for y_1''', y_2''', andy_3''' from the differential equation; multiply the first row by p_3, multiply the second row by p_2, and add these to the last row to obtain

$$W' = -p_1(t)W.$$

(c) Show that

$$W(y_1, y_2, y_3) = c\exp[-\int p_1(t)dt].$$

It follows that W is either always zero or nowhere zero on I. (d) Generalize this argument to the nth order equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0$$

with solutions $y_1, \ldots, y_n$. That is, establish Abel's formula,

$$W(y_1, \ldots, y_n)(t) = c\exp[-\int p_1(t)dt],$$

for this case.

Answer: (a). $W' = \begin{vmatrix} y_1' & y_2' & y_3' \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} + \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1'' & y_2'' & y_3'' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} + \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix} = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}$ ❏

(b). According the equation, we have

$$y_1''' = -(p_1(t)y_1'' + p_2(t)y_1' + p_3(t)y_1)$$

$$y_2''' = -(p_1(t)y_2'' + p_2(t)y_2' + p_3(t)y_2)$$

$$y_3''' = -(p_1(t)y_3'' + p_2(t)y_3' + p_3(t)y_3)$$

then,

$$W' = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ -(p_1(t)y_1'' + p_2(t)y_1' + p_3(t)y_1) & -(p_1(t)y_2'' + p_2(t)y_2' + p_3(t)y_2) & -(p_1(t)y_3'' + p_2(t)y_3' + p_3(t)y_3) \end{vmatrix}.$$

the first row $\times$ p_3 + the second row $\times$ p_2 + the last row $\Longrightarrow$

$$W' = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ -p_1(t)y_1'' & -p_1(t)y_2'' & -p_1(t)y_3'' \end{vmatrix} = -p_1(t)W$$

❑

(c).By (b), $\frac{dW/dt}{W} = -p_1(t) \Rightarrow \ln W = -\int p_1(t)dt + C \Rightarrow W(y_1, y_2, y_3)(t) = c\exp\left[-\int p_1(t)dt\right]$

(d).For the nth order equation

$$W = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}, \text{ and } W' = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}.$$

Substitute for $y_1^n, \cdots, y_n^n$ from the differential equation,

$$y_1^{(n)} = -[p_1(t)y_1^{n-1} + \cdots + p_n(t)y_1]$$

$$\vdots$$

$$y_n^{(n)} = -[p_1(t)y_n^{n-1} + \cdots + p_n(t)y_n]$$

like (a), row (1) $\times p_n(t) + \cdots +$ row $(n-1) \times p_2(t)+$ row (n), then

$$W' = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ -p_1(t)y_1^{(n-1)} & -p_1(t)y_2^{(n-1)} & \cdots & -p_1(t)y_n^{(n-1)} \end{vmatrix} = -p_1(t)W$$

so, $W(y_1, \cdots, y_n)(t) = c\exp\left[-\int p_1(t)dt\right]$.

❑

9. Use the method of variation of parameters to find solutions of the following ODEs

b). $y''' + y' = \sec t, \ -\frac{\pi}{2} < t < \frac{\pi}{2}$

c). $y''' - y' = \csc t, \ y(\frac{\pi}{2}) = 2, \ y'(\frac{\pi}{2}) = 1, \ y''(\frac{\pi}{2}) = -1$

Answer:

b).

The characteristic equation is

$$r^3 + r = r(r^2 + 1) = 0.$$

We get $r_1 = 0$, $r_2 = i$, $r_3 = -i$

The general solutions are

$$y(t) = c_1 + c_2 \cos t + c_3 \sin t.$$

$$\begin{aligned} \omega(t) &= \begin{vmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{vmatrix} \\ &= 1 \end{aligned}$$

$$\begin{aligned} \omega_1(t) &= \begin{vmatrix} 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 1 & -\cos t & -\sin t \end{vmatrix} \\ &= 1 \end{aligned}$$

$$\begin{aligned} \omega_2(t) &= \begin{vmatrix} 0 & 0 & \sin t \\ 0 & 0 & \cos t \\ 1 & 1 & -\sin t \end{vmatrix} \\ &= -\cos t \end{aligned}$$

$$\begin{aligned} \omega_3(t) &= \begin{vmatrix} 1 & \cos t & 0 \\ 0 & -\sin t & 0 \\ 0 & -\cos t & 1 \end{vmatrix} \\ &= -\sin t \end{aligned}$$

$$y_p = \int \sec t dt + \cos t \int (-\cos t) \sec t dt + \sin t \int (-sint) \sec t dt$$

$$= \ln(\sec t + \tan t) - t\cos t + \sin t \ln\cos t$$

Hence, the general solutions are

$$y(t) = y(t) = c_1 + c_2\cos t + c_3 \sin t \ln(\sec t + \tan t) - t\cos t + \sin t \ln\cos t.$$

c).

The characteristic equation is

$$r^3 - r = r(r^2 - 1) = 0.$$

We get $r_1 = 0$, $r_2 = 1$, $r_3 = -1$

The general solutions are

$$y(t) = c_1 + c_2 e^t + c_3 e^{-t}.$$

$$\begin{aligned} \omega(t) &= \begin{vmatrix} 1 & e^t & e^{-t} \\ 0 & e^t & -e^{-t} \\ 0 & e^t & e^{-t} \end{vmatrix} \\ &= 2 \end{aligned}$$

$$\begin{aligned} \omega_1(t) &= \begin{vmatrix} 0 & e^t & e^{-t} \\ 0 & e^t & -e^{-t} \\ 1 & e^t & e^{-t} \end{vmatrix} \\ &= -2 \end{aligned}$$

$$\begin{aligned} \omega_2(t) &= \begin{vmatrix} 1 & 0 & e^{-t} \\ 0 & 0 & -e^{-t} \\ 0 & 1 & e^{-t} \end{vmatrix} \\ &= e^{-t} \end{aligned}$$

$$\begin{aligned} \omega_3(t) &= \begin{vmatrix} 1 & e^t & 0 \\ 0 & e^t & 0 \\ 0 & e^t & 1 \end{vmatrix} \\ &= e^t \end{aligned}$$

$$y_p = -\int \sec t dt + \frac{1}{2}e^t \int e^{-t} \csc t dt + \frac{1}{2}e^t \int e^t \csc t dt$$
$$= -\ln \sin t + \ln(\cos t + 1) + \frac{1}{2}e^t \int e^{-t} \csc t dt + \frac{1}{2}e^t \int e^t \csc t dt$$

Hence, the general solutions are

$$y(t) = c_1 + c_2 e^t + c_3 e^{-t} - \ln \sin t + \ln(\cos t + 1) + \frac{1}{2}e^t \int e^{-t} \csc t dt + \frac{1}{2}e^t \int e^t \csc t dt.$$

From $y(\frac{\pi}{2}) = 2,\ y'(\frac{\pi}{2}) = 1,\ y''(\frac{\pi}{2}) = -1$, we get $c_1 = 3,\ c_2 = 0,\ c_3 = e^{-\frac{\pi}{2}}$

The solution of the problem is

$$y(t) = 3 + e^{-\frac{\pi}{2}} e^{-t} - \ln \sin t + \ln(\cos t + 1) + \frac{1}{2}e^t \int e^{-t} \csc t dt + \frac{1}{2}e^t \int e^t \csc t dt.$$

❑

13. Given that x, x^2, and $1/x$ are solutions of the homogeneous equation corresponding to

$$x^3 y''' + x^2 y'' - 2xy' + 2y = 2x^4, \qquad x > 0$$

determine a particular solution.

Answer: The original ODE can be written as

$$y''' + \frac{1}{x}y'' - \frac{2}{x^2}y' + \frac{2}{x^3}y = 2x$$

By method of variation of parameters, a particular solution is

$$Y(t) = \sum_{m=1}^{3} y_m(t) \int^t \frac{W_m(s)}{W(s)} ds$$

Where $y_1 = x, y_2 = x^2, y_3 = \frac{1}{x}$. By direct computation,

$$W = \begin{vmatrix} x & x^2 & \frac{1}{x} \\ 1 & 2x & -\frac{1}{x^2} \\ 0 & 2 & \frac{2}{x^3} \end{vmatrix} = \frac{6}{x} \quad W_1 = \begin{vmatrix} 0 & x^2 & \frac{1}{x} \\ 0 & 2x & -\frac{1}{x^2} \\ 1 & 2 & \frac{2}{x^3} \end{vmatrix} = -3$$

$$W_2 = \begin{vmatrix} x & 0 & \frac{1}{x} \\ 1 & 0 & -\frac{1}{x^2} \\ 0 & 1 & \frac{2}{x^3} \end{vmatrix} = \frac{2}{x} \quad W_3 = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = x^2$$

hence

$$Y(x) = x \int^x 2s(-\frac{1}{2}s) ds + x^2 \int^x 2s(\frac{1}{3}) ds + \frac{1}{x} \int^x 2s(\frac{x^3}{6}) ds = \frac{1}{15}x^4$$

Then a particular solution is

$$Y(t) = \frac{1}{15}x^4.$$

❑